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The Variance in Palmgren-Miner Damage Due To Random Vibration

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ABSTRACT

A random stress-history which is proportional to the stationary response of a single-degree-of-freedom vibratory system to wide-band Gaussian excitation is assigned a damage based on the Palmgren-Miner hypothesis and an idealized S-N law. The damage accumulated in time T is a random variable because of the randomness in the number of "cycles" and the randomness in the amplitudes of the "cycles." The mean and variance of the damage are obtained by two procedures: one which accounts for both sources of randomness and one which neglects the randomness in the number of "cycles" contained in the interval. The two procedures give the same asymptotic result when the bandwidth shrinks to zero. The theoretical results are illustrated by curves computed for a particular example.

The Variance in Palmgren-Miner Damage
Due to Random Vibration

A damage $D(T)$ can be associated with an interval T of a stationary narrow-band random stress-history $s(t)$ by using the Palmgren-Miner criterion [1]. This damage is a random variable taking on different values for each sample stress-history. In 1954 Miles [2] evaluated the expected value or mean of the damage when the stress-history was the response of a single-degree-of-freedom vibratory system to white Gaussian excitation and the S-N diagram or fatigue "law" for the material was assumed to have the form

$$NS^{\alpha} = \text{constant} = S_1^{\alpha}$$

We shall be concerned with the same situation and will evaluate the <u>variance</u> of the damage in addition to the <u>mean.</u>

The damage $D(T)$ when T is large is the sum of a large number of incremental damages each associated with a single "cycle." The randomness in D is due to the randomness in the amplitudes of the individual "cycles" and also due to the randomness in the number of "cycles" contained in the interval T. We have made two evaluations of the variance; the first takes into account both sources of randomness while the second is an approximate solution which considers only the randomness in the amplitudes and neglects the randomness in the periods of the "cycles." The two solutions are shown to approach one another in the limit as the bandwidth is decreased to zero. In both cases the major difficulty in the analysis is due to the strong correlation in the incremental damages of succeeding "cycles." It has been necessary to make approximations which are only valid when T is long in comparison with the decay time of the correlation.

It does not appear possible to obtain the complete probability distribution of the damage D(T) for finite T although the central limit theorem can be invoked to show that in the limit as $T \to \infty$ then the distribution of D becomes normal. In this limiting situation the mean and the variance are sufficient to completely characterize the distribution of D.

1. Mean and Variance of Sums

Let the total interval T be divided into M equal subintervals. With each subinterval let an incremental damage d_i be associated. For the moment we postpone the discussion of how the incremental damage is to be associated. The total damage

$$D = \sum_{i=0}^{M-1} d_i \tag{2}$$

is the sum of (correlated) random variables. The mean and variance of D are

$$\begin{aligned} E[D] &= \sum_{i=0}^{M-1} E[d_i] \\ var[D] &= E[D^2] - (E[D])^2 = \sum_i \sum_j E[d_i d_j] - (E[D])^2 \end{aligned} \tag{3}$$

Now since the stress-history is stationary the damage process is also and the statistical averages needed in (3) are invariant with respect to a translation of the time axis. Thus

$$\begin{aligned} E[d_i] &= E[d_j] = E[d_0] \\ E[d_i d_j] &= E[d_0 d_{j-i}] \end{aligned} \tag{4}$$

for arbitrary i and j. The sums in (3) may then be recast as follows

$$\begin{aligned} E[D] &= M\,E[d_o] \\ \mathrm{var}[D] &= M\{E[d_o^2]-(E[d_o])^2\}+2\sum_{k=1}^{M-1}(M-k)\{E[d_o d_k]-(E[d_o])^2\} \end{aligned} \tag{5}$$

Although the damages in adjacent subintervals may be strongly correlated d_k and d_o become uncorrelated when k gets large enough and thus

$$\lim_{k\to\infty}\{E[d_o d_k]-(E[d_o])^2\} = 0 \tag{6}$$

This will be helpful for evaluating (5) for large M.

2. Incremental damage associated with subintervals.

Ordinarily the Palmgren-Miner hypothesis is used to associate a damage

$$d_i = \frac{1}{N_i} \tag{7}$$

with the i-th cycle where N_i is the number of cycles until failure at the constant stress amplitude S_i as given by the S-N diagram or by a relation such as (1). In order to avoid certain subtleties involved in determining the peak amplitudes of a random process we consider a slight modification of the hypothesis in which we associate a damage

$$d = \frac{1}{2N} \tag{8}$$

with a zero-crossing of the stress process using the slope $\dot{s}$ as a measure of the stress-amplitude. If the expected frequency of the narrow band process is ω_o where

$$\omega_o^2 = \frac{\sigma_{\dot{s}}^2}{\sigma_s^2} = \frac{-R''(0)}{R(0)} = \frac{\int_{-\infty}^{\infty}\omega^2 G(\omega)\,d\omega}{\int_{-\infty}^{\infty}G(\omega)\,d\omega} \tag{9}$$

and $R(\tau)$ and $G(\omega)$ are the autocorrelation function and spectral density respectively of the process (assumed to have zero mean) then the equivalent stress amplitude we associate with a zero-crossing having slope $\dot{s}$ is

$$S = \frac{|\dot{s}|}{\omega_o} \tag{10}$$

and the number N in (8) is obtained from the S-N diagram using (10) for S. The factor 2 in (8) arises because there are twice as many zero-crossings as cycles in a narrow-band process; it could be avoided by considering only the zero-crossings with positive slope but the integrations leading to (21) are considerably simpler when we associate damage with half-cycles rather than with cycles. The relation (10) would be strictly correct for simple harmonic motion at frequency ω_o. In a narrow-band process it represents a good approximation to the amplitude of the stress peak immediately after (or before) the zero-crossing. We believe that the statistics of the zero-crossing damage process so defined will not differ significantly from the statistics of the peak-associated damage process.

The two evaluations which follow are based on two choices for the subinterval duration. In the first case the subinterval is taken to have the duration Δt and eventually Δt is taken to approach zero. This procedure permits us to take into account the variation in the periods of the "cycles." In this case the incremental damage d_i is taken to be <u>zero</u> if there is no zero-crossing within the subinterval or to be the value (8) if there is a zero-crossing. In the second case the variation in the periods of the "cycles" is neglected and the duration of each subinterval is taken to be π/ω_o ; i.e., the expected duration of a half-cycle. The incremental damage is taken to be (8) but here instead of using (10) we find it more convenient to use the value of S given by Rice's envelope function [3].

3. First case: infinitesimal subintervals.

The interval T is divided into equal subintervals Δt such that

$$M \Delta t = T \tag{11}$$

We assume that Δt is so small that the stress-history s(t) can be taken as a straight line throughout the interval. The fraction of samples which will have a zero-crossing in a particular subinterval Δt can be ascertained by considering the distribution of combinations s(t) and $\dot{s}(t)$ where t is the time at the beginning of the interval. This distribution is described by the joint density function $p(s,\dot{s})$. Combinations of s and $\dot{s}$ which involve a zero-crossing are those for which

$$\begin{aligned} -\dot{s}\Delta t \leq s < 0, \qquad & \dot{s} > 0 \\ 0 < s < -\dot{s}\Delta t, \qquad & \dot{s} < 0 \end{aligned} \tag{12}$$

For those samples having a zero-crossing the incremental damage is given by (8); for those without a zero-crossing there is no damage. The expected damage in the subinterval Δt is then

$$\begin{aligned} E[d_o] &= \int_{-\infty}^{0} \frac{1}{2N}\, d\dot{s} \int_{0}^{-\dot{s}\Delta t} p(s,\dot{s})\, ds + \int_{0}^{\infty} \frac{1}{2N}\, d\dot{s} \int_{-\dot{s}\Delta t}^{0} p(s,\dot{s})\, ds \\ &= \Delta t \int_{-\infty}^{\infty} \frac{1}{2N} |\dot{s}|\, p(0,\dot{s})\, d\dot{s} \end{aligned} \tag{13}$$

To evaluate (13) we make the assumption that s(t) is a Gaussian process so that

$$p(0,\dot{s}) = \frac{1}{\sqrt{2\pi}\,\sigma_{\dot{s}}} e^{-\frac{\dot{s}^2}{2\sigma_{\dot{s}}^2}} \tag{14}$$

and we assume that (1) is the S-N curve of the material so that using (10)

$$\frac{1}{2N} = \frac{1}{2}\left(\frac{|\dot{s}|}{\omega_0 S_1}\right)^{\alpha} \tag{15}$$

Inserting (14) and (15) into (13) yields

$$E[d_0] = \Delta t \; \frac{\omega_0}{2\pi}\left(\frac{\sqrt{2}\,\sigma_s}{S_1}\right)^{\alpha}\Gamma(1+\alpha/2) \tag{16}$$

Finally inserting (16) and (11) into (5) leads to the mean or expected damage for an interval T

$$E[D(T)] = \nu_0^+ T\left(\frac{\sqrt{2}\,\sigma_s}{S_1}\right)^{\alpha}\Gamma(1+\alpha/2) \tag{17}$$

where ν_0^+ is the expected number of zero-crossings with positive slope; i.e., ν_0^+ is the expected or mean frequency in cycles per unit time

$$\nu_0^+ = \frac{\omega_0}{2\pi} \tag{18}$$

The result (17) although derived differently is identical with that of Miles [2].

The above derivation can be repeated using the square of (15) to obtain the mean square damage associated with the subinterval Δt.

$$E[d_s^2] = \frac{\nu_o^+ \Delta t}{2} \left(\frac{2\sigma_s^2}{S_1^2} \right)^{\alpha} \Gamma(1+\alpha) \tag{18}$$

The derivation for the correlation terms $E[d_o d_k]$ is similar but more complex. The product $d_o d_k$ is zero unless zero-crossings occur in <u>both</u> the zero-th and k-th subintervals in which case the product is given by multiplying together terms of the form (8) which in turn depend (10) on the slopes $\dot{s}$ at the zero-crossings. The fraction of samples which will have zero crossings in both subintervals can be ascertained by considering the distribution of combinations $s(t)$, $\dot{s}(t)$, $s(t + k\Delta t)$ and $\dot{s}(t + k\Delta t)$ which is described by the four-dimensional joint probability density $p(s(t), \dot{s}(t), s(t+\tau), \dot{s}(t+\tau))$. Again by integrating over the subregion where both subintervals have zero-crossings we find, analogous to (13)

$$E[d_o d_k] = (\Delta t)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2N_o} \frac{1}{2N_k} |\dot{s}_o| |\dot{s}_k| p(0, \dot{s}_o, 0, \dot{s}_k) d\dot{s}_o d\dot{s}_k \tag{19}$$

for $k \neq 0$. To evaluate (19) we make the assumption that $s(t)$ is a Gaussian process so that [3]

$$p(0, \dot{s}_o, 0, \dot{s}_k) = \frac{1}{4\pi^2 \Lambda^{1/2}} \exp\left[-\frac{1}{2\Lambda} \left(\Lambda_{22} \dot{s}_o^2 + 2\Lambda_{24} \dot{s}_o \dot{s}_k + \Lambda_{44} \dot{s}_k^2 \right) \right] \tag{20}$$

where the Λ-parameters depend on the autocorrelation of the stress process and are evaluated in Sec. 4. Again we assume that the incremental damages are given by (15) but here to render the integral tractable we make the additional assumption that the exponent α is an odd positive integer. In this way we find [4]

$$E[d_o d_k] = \left(\frac{\Delta t}{2\pi}\right)^2 \left(\frac{2}{S_1 \omega_o^2}\right)^{\alpha} \Gamma^2(1+\alpha/2) \frac{\Lambda^{\alpha+3/2}}{\Lambda_{22}^{\alpha+2}} F\left(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; \frac{\Lambda_{24}^2}{\Lambda_{22}^2}\right) \tag{21}$$

Finally to obtain the variance in the total damage we insert (21), (18) and (16) into (5). At this time we also let $\Delta t \to 0$ thereby converting the summation into a Riemann integral. With $k\Delta t = \tau$ we find

$$var[D(T)] = \frac{\nu_o^+ T}{2}\left(\frac{2\sigma_S^2}{S_1^2}\right)^{\alpha} \Gamma(1+\alpha) +$$

$$2\left(\frac{2\sigma_S^2}{S_1^2}\right)^{\alpha}\left(\frac{\Gamma(1+\frac{\alpha}{2})}{2\pi}\right)^2 \int_0^T \omega_o^2 (T-\tau)\left\{\frac{\Lambda^{\alpha+3/2}}{\omega_o^{2\alpha+2}\sigma_S^{2\alpha}\Lambda_{22}^{\alpha+2}} F\left(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; \frac{\Lambda_{24}^2}{\Lambda_{22}^2}\right) - 1\right\} d\tau \tag{22}$$

Further evaluation requires a specific choice for the autocorrelation function $R(\tau)$ in order to specify the Λ-parameters and the hypergeometric function.

4. Specialization to the response of a single-degree-of-feedom system.

We limit our discussion to the case where the stress history $s(t)$ is proportional to the response of a lightly-damped single-degree-of-freedom oscillator when excited by stationary white noise; i.e., $s(t)$ is taken to satisfy the differential equation

$$\ddot{s} + 2\zeta\omega_n \dot{s} + \omega_n^2 s = f(t) \tag{23}$$

where ζ is the damping ratio and ω_n is the undamped natural frequency. When the excitation $f(t)$ is stationary white noise the autocorrelation function of the response is

$$R(\tau) = \sigma_\varsigma^2 e^{-\varsigma\omega_n\tau}\left(\cos p\tau + \frac{\varsigma\omega_n}{p}\sin p\tau\right), \quad \tau \geq 0 \tag{24}$$

where $p = \sqrt{1-\zeta^2}\,\omega_n$ is the damped natural frequency and σ_ς^2 is the mean square stress. The Λ-parameters are obtained from $R(\tau)$ according to the following definitions

$$\Lambda = \left[R(0)^2 - R(\tau)^2\right]\left[R''(0)^2 - R''(\tau)^2\right] + 2R'(\tau)^2\left[R(0)R''(0) - R(\tau)R''(\tau)\right] + R'(\tau)^4$$

$$\Lambda_{22} = \Lambda_{44} = -R''(0)\left[R(0)^2 - R(\tau)^2\right] - R(0)R'(\tau)^2 \tag{25}$$

$$\Lambda_{24} = R''(\tau)\left[R(0)^2 - R(\tau)^2\right] + R(\tau)R'(\tau)^2$$

Using (24) we find that the expected frequency of the narrow band process, ω_0 of (9), is just the natural frequency ω_n. It is thus possible in principle to insert (25) and (24) in (22) and evaluate the variance of the damage. The integration appears however to involve formidable difficulties. We therefore discontinue the exact evaluation and consider an approximation for small damping. We use the following small ζ approximations to (25) for use in (22)

$$\Lambda = \sigma_\varsigma^8\omega_n^4\left(1 - e^{-2\varsigma\omega_n\tau}\right)^2$$

$$\Lambda_{22} = \sigma_\varsigma^6\omega_n^2\left(1 - e^{-2\zeta\omega_n\tau}\right)$$

$$\Lambda_{24} = -\sigma_\varsigma^6\omega_n^2\left(1 - e^{-2\varsigma\omega_n\tau}\right)e^{-\varsigma\omega_n\tau}\cos\omega_n\tau \tag{26}$$

$$\frac{\Lambda^{\alpha+1/2}}{\omega_n^{2+2\alpha}\sigma_\varsigma^{2\alpha}\Lambda_{22}^{\alpha}} = \left(1 - e^{-2\varsigma\omega_n\tau}\right)^{1+\alpha}$$

$$\frac{\Lambda_{24}^2}{\Lambda_{22}^2} = e^{-2\varsigma\omega_n\tau}\cos^2\omega_n\tau$$

The argument of the hypergeometric function thus oscillates with period π/ω_n. The integrand in (22) has peaks at $n\pi/\omega_n$ and falls to zero at $(2n+1)\pi/2\omega_n$, $n = 1,2,\ldots$. Furthermore the integrand is negligible in the range $0 < \tau < \pi/2\omega_n$. We have been able to obtain a good approximation to the integral [4] by substituting a smoothed integrand in which the expression in braces in (22) is replaced by

$$\left\{ F\left(-\tfrac{\alpha}{2}, -\tfrac{\alpha}{2}; 1; e^{-2\zeta\omega_n\tau}\right) - 1 \right\} \tag{27}$$

and the limits of integration are changed to $\pi/2\omega_n$ and ∞. The approximations involved are good for $\zeta << 1$ and $\zeta\omega_n T >> 1$. The result is

$$\operatorname{var}[D(T)] = \frac{\nu_0^+ T}{\zeta}\left(\frac{2\sigma_S^2}{S_1^2}\right)^{\alpha}\Gamma^2\left(1+\frac{\alpha}{2}\right)\left\{f_1(\alpha) - \frac{f_2(\alpha)}{\zeta\nu_0^+ T} + \frac{\zeta f_3(\alpha)}{\nu_0^+ T}\right\} \tag{28}$$

where the f-quantities are given by

$$\begin{aligned} f_1(\alpha) &= \frac{\zeta}{\pi}\int_0^{\infty}\left\{F\left(-\tfrac{\alpha}{2}, -\tfrac{\alpha}{2}; 1; e^{-\zeta\omega_n\tau}\right) - 1\right\}\omega_n d\tau \\ f_2(\alpha) &= \frac{\zeta^2}{2\pi^2}\int_0^{\infty}\omega_n\tau\left\{F\left(-\tfrac{\alpha}{2}, -\tfrac{\alpha}{2}; 1; e^{-\zeta\omega_n\tau}\right) - 1\right\}\omega_n d\tau \\ f_3(\alpha) &= \frac{\Gamma(1+\alpha)}{16\,\Gamma^2(1+\alpha/2)} \end{aligned} \tag{29}$$

and are tabulated in Table I. A partial check on the accuracy of approximation was made by comparing the smoothed integral used above with the results of a numerical integration of the exact integral. For $\alpha = 9$ and $\zeta = 1/60$ the discrepancy was less than one part in 300. The errors tend to increase rapidly with α but for the range tabulated in Table I the result (28) is probably satisfactory for engineering purposes for $\zeta \leq 0.05$ ($Q > 10$) if the expected number of cycles $\nu_0^+ T$ is very large compared with $Q = 1/2\zeta$. In this range very little additional error is made by neglecting the f_2 and f_3 terms in comparison with the f_1 term in (28). Thus in the range indicated we may use

$$Var\left[D(T)\right] = \frac{\nu_0^+ T}{\zeta}\left(\frac{2\sigma_s^2}{S_1^2}\right)^{\alpha}\Gamma^2\left(1+\frac{\alpha}{2}\right) f_1(\alpha) \tag{30}$$

which indicates a linear growth of the variance with T and an inverse dependence on the damping ratio ζ for a given material and a fixed mean square stress level.

If we denote the standard deviation of the total damage by σ_D we can combine the results (17) and (30) for the mean and variance into the following ratio

$$\frac{\sigma_D}{E[D]} = \left(\frac{f_1(\alpha)}{\zeta \nu_0^+ T}\right)^{1/2} \tag{31}$$

which indicates that the <u>relative</u> variance for a given material decreases in inverse proportion to the square root of the product of T and the bandwidth $2\zeta\nu_0^+$ (or ν_0^+/Q).

5. Second case: half-cycle subintervals.

Here we consider the stress process s(t) to consist of a sequence of half-cycles each of duration π/ω_0; i.e., we neglect the random variation in periods. With each half-cycle the incremental damage (8) is 1/2N where the S-N diagram or (1) is used to relate N to a stress amplitude S associated with the half-cycle. For this purpose we find it convenient to use Rice's envelope function S(t). For a narrow-band process Rice [3] has shown that the first and second order probability densities for S(t) are

$$p(S) = \frac{S}{\sigma_s^2} e^{-\frac{S^2}{2\sigma_s^2}} \quad , \quad S > 0$$

$$p(S_0, S_k) = \frac{S_0 S_k}{A} I_0\left(\frac{B}{A} S_0 S_k\right) e^{-\frac{\sigma_s^2}{2A}(S_0^2 + S_k^2)} \tag{32}$$

where σ_s^2 is the mean square of the process s(t) and the quantities $A = \sigma_s^4 - B^2$ and B are functions of the time interval $t_k - t_0$ and depend on the spectral density of s(t). They will be described later.

The expected value of the incremental damage is

$$E[d_0] = \int_0^\infty \frac{1}{2N(S)} p(S)\, dS \tag{33}$$

Substituting from (1) and (32) yields

$$E[d_0] = \frac{1}{2} \left(\frac{\sqrt{2}\,\sigma_s}{S_1}\right)^\alpha \Gamma(1 + \alpha/2) \tag{34}$$

Note that this is equivalent to (16) with $\Delta t = \pi/\omega_0$. This is essentially the technique used by Miles [2] in deriving (17). An exactly similar evaluation using $(1/2N)^2$ in place of 1/2N in (33) leads to

$$E[d_0^2] = \frac{1}{4} \left(\frac{2\sigma_s^2}{S_1^2}\right) \Gamma(1 + \alpha) \tag{35}$$

which is equivalent to (18) with $t = \pi/\omega_0$.

Turning next to the correlation terms in (5) we have

$$E\,[d_0 d_k] = \int_0^{\infty}\int_0^{\infty} \frac{1}{2N_0}\,\frac{1}{2N_k}\; p(S_0, S_k)\, dS_0\, dS_k \tag{36}$$

which on substitution from (1) and (32) yields

$$E\,[d_0 d_k] = \frac{1}{4}\left(\frac{2\sigma_S^{\,2}}{S_1^{\,2}}\right)^{\alpha} \Gamma^2\!\left(1+\frac{\alpha}{2}\right) F\!\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; \frac{B^2}{\sigma_S^{\,4}}\right) \tag{37}$$

which is now quite different from (21) because (37) is the correlation between two half-cycles separated by $k\pi/\omega_0$ while (21) is the correlation between two infinitesimal intervals separated by $k\Delta t$. Finally to obtain the variance we insert (37), (35) and (34) into (5) to obtain the variance after M half-cycles ($T = M\pi/\omega_0 = M\,\nu_0^+/2$)

$$\begin{aligned} var\,[D(T)] = {} & \frac{\nu_0^+ T}{2}\left(\frac{2\sigma_S^{\,2}}{S_1^{\,2}}\right)^{\alpha}\left[\Gamma(1+\alpha) - \Gamma^2(1+\alpha/2)\right] + \\ & \frac{1}{2}\left(\frac{2\sigma_S^{\,2}}{S_1^{\,2}}\right)\Gamma^2\!\left(1+\frac{\alpha}{2}\right)\sum_{k=1}^{M-1}(M-k)\left\{F\!\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; \frac{B^2}{\sigma_S^{\,4}}\right) - 1\right\} \end{aligned} \tag{38}$$

Further evaluation requires a specific choice for the spectral density $G(\omega)$ of the process in order to specify the parameter B and hence the hypergeometric function.

6. Specialization to the response of a single-degree-of-feedom system .

We limit our discussion again to the process described in Sec. 4. The spectral density of the process is

$$G(\omega) = \frac{G_0}{(\omega^2 - \omega_n^2) + 4\zeta^2\omega_n^2\omega^2} \tag{39}$$

and the parameter B follows from the following operation

$$\begin{aligned} B^2 &= \mu_{13}^2 + \mu_{14}^2 \\ \mu_{13} &= \int_0^\infty G(\omega)\cos(\omega-\omega_n)\tau\, d\omega \\ \mu_{14} &= \int_0^\infty G(\omega)\sin(\omega-\omega_n)\tau\, d\omega \end{aligned} \tag{40}$$

This is quite complicated in general, but here we are only interested in times τ of the form $k\pi/\omega_n$ and moreover we will again accept the same type of small damping approximations used in Sec. 4. Under these circumstances we find simply

$$B_k = \sigma_S^2 e^{-k\pi\zeta} \tag{41}$$

and the term in the braces being summed in (38) becomes

$$\left\{ F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; e^{-2k\pi\zeta}\right) - 1 \right\} \tag{42}$$

At this stage there is considerable similarity between the term being summed (42) and the smoothed integrand term (27). In fact the sum may be considered as a crude attempt to approximate the integral. The hypergeometric functions in (38) can be expanded in series and the order of summations interchanged. The k-summations involve simple geometric progressions which can be summed. When M is large and ζ is small the resulting series can be recognized as equivalent to the functions defined in (29). In this way we obtain

$$var[D(T)] = \left(\frac{2\sigma_S^2}{S_1^2}\right)^{\alpha}\Gamma^2\left(1+\frac{\alpha}{2}\right)\left\{\frac{\nu_0^+T}{\zeta}\left[f_1(\alpha) - 8\zeta f_3(\alpha) + \frac{\zeta}{2}\right] - \frac{f_2(\alpha)}{\zeta^2} + \frac{f_1(\alpha)}{2\zeta}\right\} \tag{43}$$

for the variance when ζ is small and the expected number of cycles ν_0^+T is large. Again if $\zeta\nu_0^+T$ is large (even though ζ is small) we can dispense with the final two terms and use the asymptotic form

$$var[D(T)] = \frac{\nu_0^+T}{\zeta}\left(\frac{2\sigma_S^2}{S_1^2}\right)^{\alpha}\Gamma^2\left(1+\frac{\alpha}{2}\right)\left\{f_1(\alpha) - \zeta\left[8f_3(\alpha) - \frac{1}{2}\right]\right\} \tag{44}$$

which should be compared with (30). We note that the discrepancy between the first case which accounted for the random variations in periods and the second case which neglected this source of variance is small for light damping and that the ratio of the two expressions (30) and (44) approaches unity as $\zeta \to 0$.

7. Example

Consider the system shown in Fig. 1 in which the vehicle has a stationary random acceleration with uniform spectral density 0.5 g^2/cps. With the following data

L,	length of cantilever,	$= 4.0''$
h,	side of square cross section	$= 0.25''$
E,	75S-T6 aluminum alloy	$= 10.3 \times 10^6$ psi
α,	" " "	$= 6.09$
S_1,	" " "	$= 2 \times 10^5$ psi
m,	mass of one cubic inch of steel	$= 7.28 \times 10^4$ lb sec^2/in

the natural frequency of the system is

$$\omega_n = 465 \text{ rad/sec} \quad (73.9 \text{ cps})$$

and the rms stress level of the narrow-band stress response in the extreme fibers at the root of the cantilever is

$$\sigma_S = \frac{2320}{\sqrt{\zeta}} \text{ psi} \tag{45}$$

where ζ is the damping ratio of the system. With these values it is now possible to find the expected damage (17) and the standard deviation of the damage (31) as functions of ζ and T. The value 2.32 for $f_1(\alpha)$ is obtained from interpolation in Table I. The results are shown in Fig. 2 for four different values of system damping. The expected damages appear as straight lines with unit slopes indicating linear growth with time. The variance in the accumulated damage is suggested by the curves showing $E[D] \pm \sigma_D$. Note here that the primary effect of the damping is through its action in setting the rms stress level (45) while a secondary effect is its action in controlling the deviation of the damage through the correlation of the damages of successive cycles (31). It is also of interest to compare the two solutions (30) and (44) for the variance in the range covered by Fig. 2. We find a 25% discrepancy in the variance for $Q = 10$ and 2.5% for $Q = 100$. The discrepancies in the deviation σ_D would be about half of these values.

The Palmgren-Miner criterion for failure is $D = 1$. Fig. 2 indicates that when the average damage reaches unity there is actually a distribution of damage across the ensemble of sample histories. There will correspondingly be a distribution of time-to-failure T_F. This distribution is unknown but the central limit theorem can be invoked to show that it also becomes asymptotically Gaussian as $T \to \infty$ and that the plus and minus one-sigma limits for T_F are asymptotically the points where the plus and minus one-sigma curves for damage cross $D = 1$.

Table 1 Quantities in Equation (28) Which Depend on the Fatigue Law Exponent α.

	$f_1(\alpha)$	$f_2(\alpha)$	$f_3(\alpha)$
1	0.0414	0.00323	0.0796
3	0.369	0.0290	0.212
5	1.280	0.0904	0.679
7	3.72	0.223	2.33
9	10.7	0.518	8.28
11	31.5	1.230	30.0
13	96.7	3.06	111.2
15	308.	8.11	415.

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CAPTIONS FOR FIGURES

Fig. 1 (a) Random stress history at A is due to random vibration of vehicle. (b) Schematic excitation - response diagram.

Fig. 2 Palmgren-Miner damage at root of cantilever beam. The mean damage expected is shown together with the plus and minus one-sigma limits.

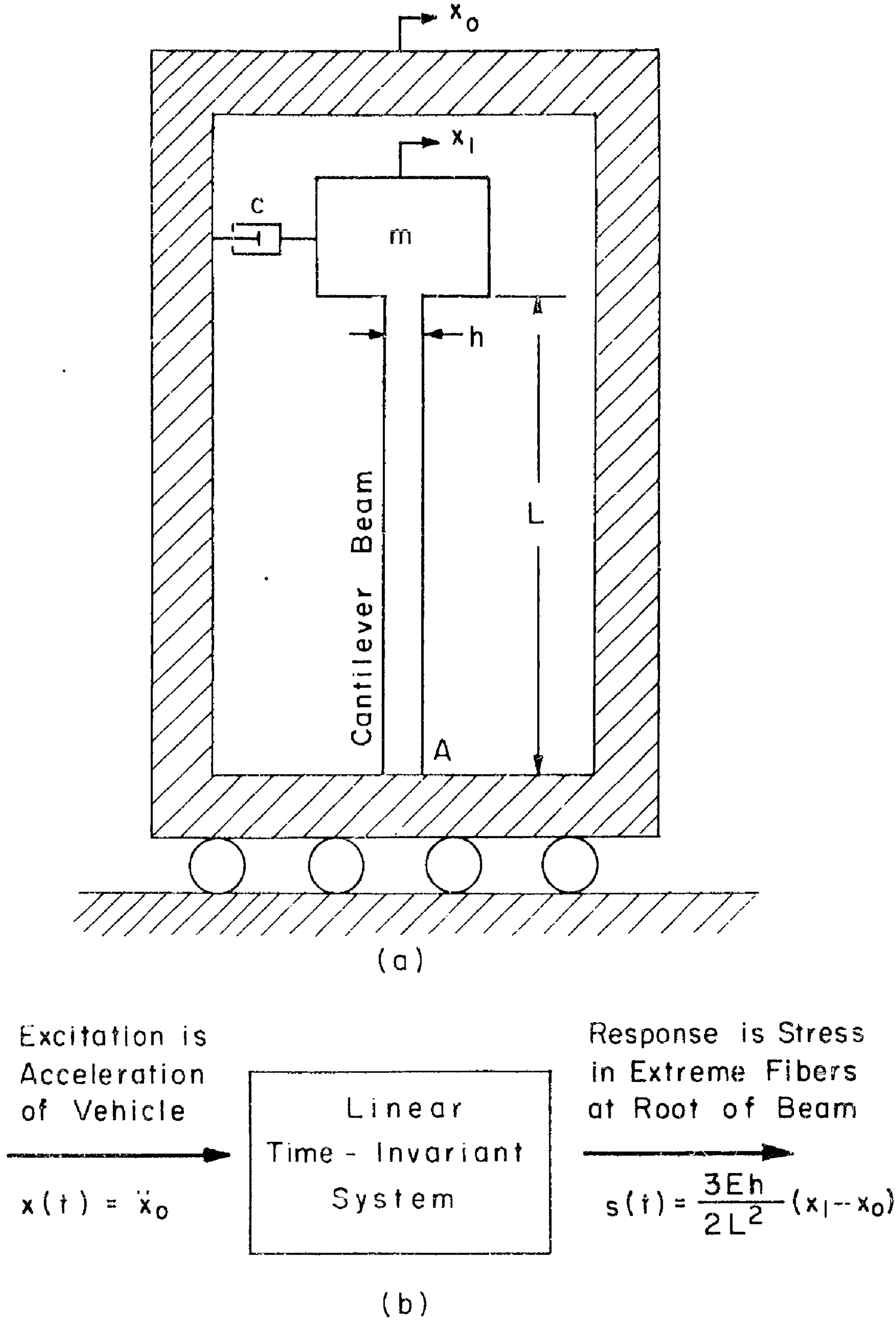
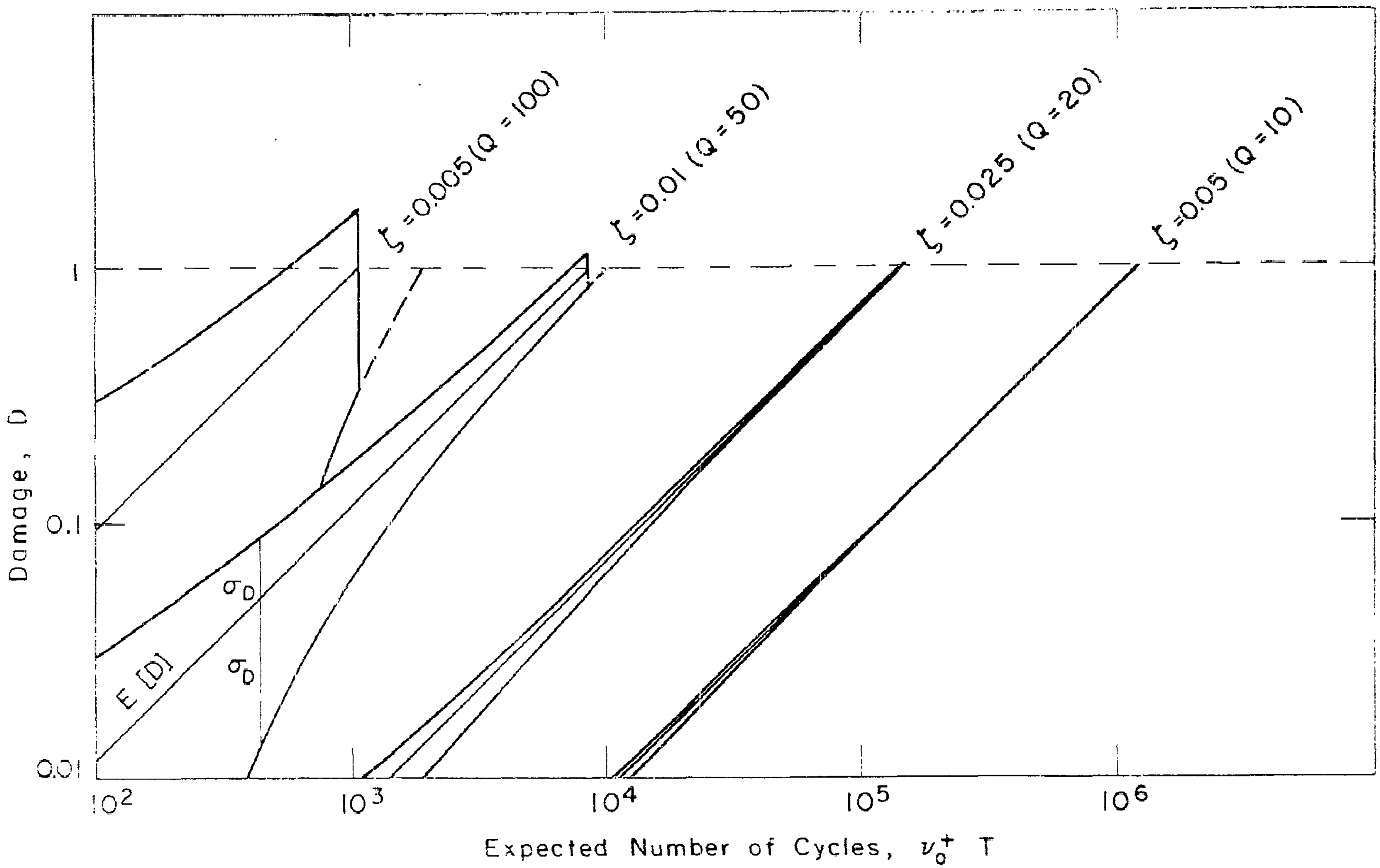

x_o
x_1
c
m
h
L
Cantilever Beam
A
(a)
Excitation is Acceleration of Vehicle
$x(t) = \ddot{x}_o$
Linear Time - Invariant System
Response is Stress in Extreme Fibers at Root of Beam
$s(t) = \frac{3Eh}{2L^2}(x_1 - x_o)$
(b)

FIGURE 1



Damage, D
1
0.1
0.01
10^2
10^3
10^4
10^5
10^6
Expected Number of Cycles, $\nu_0^+ T$
ζ =0.005 (Q = 100)
ζ =0.01 (Q = 50)
ζ =0.025 (Q = 20)
ζ =0.05 (Q = 10)
E [D]
σ_D
σ_D

FIGURE 2